All topologies on a set with two or three elements

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In this paper, we find all possible topologies on a set of two or three elements.

1. Prelude

For the purpose of our enumeration, we first give a characterization of topologies on finite sets.

Proposition 1.

Let X be a finite set. A subset $\mathfrak{O}$ of $\mathscr{P}(X)$ is a topology on X if and only if the following conditions are satisfied:

(1) $\emptyset$ and X belong to $\mathfrak{O}$.

(2) If A and B are sets belonging to $\mathfrak{O}\backslash\{\emptyset, X\}$, then their union $A \cup B$ belongs to $\mathfrak{O}$.

(3) If A and B are sets belonging to $\mathfrak{O}\backslash\{\emptyset, X\}$, then their intersection $A \cap B$ belongs to $\mathfrak{O}$.

Proof :

Clearly, if $\mathfrak{O}$ is a topology on X, the conditions **(1)**, **(2)** and **(3)** hold.

Conversely, we assume that the conditions **(1)**, **(2)** and **(3)** are satisfied. Let $\mathfrak{A}$ be a subset of $\mathfrak{O}$. Then $\bigcup \mathfrak{A} = X$ if $X \in \mathfrak{A}$, while $\bigcup \mathfrak{A} = \emptyset$, if $\mathfrak{A} = \emptyset$ or $\mathfrak{A} = \{\emptyset\}$; otherwise, since $\mathscr{P}(X)$ is finite, the union $\bigcup \mathfrak{A}$ can be written as a finite union of sets of $\mathfrak{O}$. By **(1)** and **(2)**, it follows that every union of sets of $\mathfrak{O}$ is a set of $\mathfrak{O}$. Moreover, the intersection of two sets of $\mathfrak{O}$ is $\emptyset$ if one of the two sets is the empty set, or one of the two sets if the other is equal to X; otherwise, it is the intersection of two sets belonging to $\mathfrak{O}\backslash\{\emptyset, X\}$. Therefore, the intersection of two sets of $\mathfrak{O}$ belongs to $\mathfrak{O}$. Hence, if the conditions **(1)**, **(2)** and **(3)** are satisfied, then $\mathfrak{O}$ is a topology on X. $\square$

On any set (finite or not), apart from the discrete and the indiscrete topologies, there are other trivial topologies revealed by the following result:

Proposition 2.

Let X be a set. Then, for each subset A of X, the set $\mathfrak{O}_A = \Big\{\emptyset, A, X\Big\}$ is a topology on X. It is called the **topology generated** by A.

Proof :

We can easily establish that every union of sets of $\mathfrak{O}_A$ is a set of $\mathfrak{O}_A$, and that the intersection of two sets of $\mathfrak{O}_A$ belongs to $\mathfrak{O}_A$. This is sufficient to conclude the proof, since $\emptyset$ and X are members of $\mathfrak{O}_A$. □

2. All topologies on a two-elements set

Let $X = \{a, b\}$, where a and b are distinct. Then

$$\mathscr{P}(X) = \Big\{\emptyset, \{a\}, \{b\}, X\Big\}.$$

By definition, any topology on X is a subset of $\mathscr{P}(X)$ containing $\emptyset$ and X. There are exactly four subsets $\mathscr{P}(X)$, which contain $\emptyset$ and X; namely,

$$\Big\{\emptyset, X\Big\}, \qquad \Big\{\emptyset, \{a\}, X\Big\}, \qquad \Big\{\emptyset, \{b\}, X\Big\} \qquad \text{and} \qquad \mathscr{P}(X).$$

Each of these four subsets of $\mathscr{P}(X)$ is a topology on X; this fact follows straightforwardly from Proposition 1. Consequently, there are exactly four topologies on the set $X = \{a, b\}$: the *indiscrete* topology $\Big\{\emptyset, X\Big\}$, the *discrete* topology $\mathscr{P}(X)$, plus the topologies

$$\Big\{\emptyset, \{a\}, X\Big\} \qquad \text{and} \qquad \Big\{\emptyset, \{b\}, X\Big\},$$

generated respectively by the singletons $\{a\}$ and $\{b\}$.

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It is important to notice that, in the case of a two-elements sets X, all the subsets of $\mathscr{P}(X)$ containing $\emptyset$ and X are topologies on X, that is, only **4** among the $2^4 = 16$ subsets of $\mathscr{P}(X)$.

3. All topologies on a three-elements set

Let $X = \{a, b, c\}$, where a, b and c are pair-wise distinct. Then

$$\mathscr{P}(X) = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\Big\}.$$

There are exactly $2^6 = 64$ subsets of $\mathscr{P}(X)$, which contain $\emptyset$ and X; indeed, every subset of $\mathscr{P}(X)$, which contains $\emptyset$ and X, has the form $\Big\{\emptyset, X\Big\} \cup \mathfrak{A}$, where $\mathfrak{A}$ is a subset of $\mathscr{P}(X)\backslash\Big\{\emptyset, X\Big\}$. It would be tedious to verify which of the 64 subsets of $\mathscr{P}(X)$ is a topology on X, like we did above for a set with two elements. Fortunately, we can rule out a great deal of these 64 subsets of $\mathscr{P}(X)$ by noticing the following:

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> If a topology $\mathfrak{O}$ contains the three singletons $\{a\}$, $\{b\}$ and $\{c\}$ (resp. the three pairs $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$), then $\mathfrak{O} = \mathscr{P}(X)$.

Indeed, $\{a\} \cup \{b\} = \{a, b\}$ and $\{a\} \cup \{c\} = \{a, c\}$, while $\{b\} \cup \{c\} = \{b, c\}$. Moreover,

$$\{a, b\} \cap \{a, c\} = \{a\}, \quad \{a, b\} \cap \{b, c\} = \{b\} \quad \text{and} \quad \{a, c\} \cap \{b, c\} = \{c\}.$$

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> Therefore, apart from the indiscrete topology $\Big\{\emptyset, X\Big\}$, the discrete topology $\mathscr{P}(X)$, and the **six** topologies generated respectively by the six members of $\mathscr{P}(X)\backslash\{\emptyset, X\}$ (see Proposition 2 on page 2), any other topology on X contains exactly m singletons and n pairs, where $m \in \{1, 2\}$ and $n \in \{1, 2\}$.

For each $(m, n) \in \{1, 2\} \times \{1, 2\}$, let $\mathfrak{T}_{m,n}$ be the set of all topologies on X, which contains exactly m singletons and n pairs.

The set $\mathfrak{T}_{1,1}$ has precisely **nine** members:

$$\Big\{\emptyset, \{a\}, \{a, b\}, X\Big\}; \quad \Big\{\emptyset, \{a\}, \{a, c\}, X\Big\}; \quad \Big\{\emptyset, \{a\}, \{b, c\}, X\Big\};$$
$$\Big\{\emptyset, \{b\}, \{a, b\}, X\Big\}; \quad \Big\{\emptyset, \{b\}, \{a, c\}, X\Big\}; \quad \Big\{\emptyset, \{b\}, \{b, c\}, X\Big\};$$
$$\Big\{\emptyset, \{c\}, \{a, b\}, X\Big\}; \quad \Big\{\emptyset, \{c\}, \{a, c\}, X\Big\}; \quad \Big\{\emptyset, \{c\}, \{b, c\}, X\Big\}.$$

Clearly, if a topology on X contains exactly two singletons and one pair, then the pair is the union of the two singletons. Therefore, the set $\mathfrak{T}_{2,1}$ has precisely **three** members:

$$\Big\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\Big\}; \quad \Big\{\emptyset, \{a\}, \{c\}, \{a, c\}, X\Big\}; \quad \Big\{\emptyset, \{b\}, \{c\}, \{b, c\}, X\Big\}.$$

Any two pairs of $\mathscr{P}(X)$ has one and only one common element. Thus, if a topology on X contains exactly one singleton and two pairs, then the singleton is the intersection of the two pairs. Hence, the set $\mathfrak{T}_{1,2}$ has precisely **three** members:

$$\Big\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\Big\}; \; \Big\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\Big\}; \; \Big\{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\Big\}.$$

If a topology on X contains exactly two singletons and two pairs, then the union of the two singletons is one of the pairs, and the other pair can be any of the remaining pairs in $\mathscr{P}(X)$. Consequently, the set $\mathfrak{T}_{2,2}$ has precisely **six** members:

$$\Big\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\Big\}; \qquad \Big\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\Big\};$$

$$\Big\{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, X\Big\}; \qquad \Big\{\emptyset, \{a\}, \{c\}, \{a,c\}, \{b,c\}, X\Big\};$$

$$\Big\{\emptyset, \{b\}, \{c\}, \{a,b\}, \{b,c\}, X\Big\}; \qquad \Big\{\emptyset, \{b\}, \{c\}, \{a,c\}, \{b,c\}, X\Big\}.$$

All the topologies on the three-elements set $X = \{a, b, c\}$ are so found.

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All things considered, among the $2^8 = 256$ subsets of $\mathscr{P}(X)$, only 64 contain $\emptyset$ and X; among those 64, only $2 + 6 + 9 + 3 + 3 + 6 = \mathbf{29}$ are topologies on X.