Properties of the frontier operator

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July 23, 2021

The framework of this paper is an exercise of the volume [1] on *General Topology* by Bourbaki (cf. Exercise 5 of the section §1 in Chapter I). The paper, divided into five sections, is devoted to the *frontier operator* and some of its properties.

The first section gives the definition of the frontier and some elementary facts. The second section shows that the frontier of a set contains the frontier of its closure and the frontier of its interior, and that these three frontiers can be pairwise distinct. In the third section, we prove that frontier of the union of two sets is contained in the union of the frontiers of the two sets, and that the equality does not hold in general. The fourth section reveals a sufficient condition such that the frontier of the union of two sets is equal to the union of their frontiers. In the fifth section, we establish a property of the frontier of the intersection of two open sets.

1. Definition and elementary facts

Definition 1.

In a topological space X, a point x is said to be a **frontier point** of a set A if it lies in the closure of A and in the closure of $X\backslash A$. The set of frontier points of A, denoted by $\mathrm{Fr}(A)$, is called **frontier** of A.

From this definition, we obtain easily

$$\mathrm{Fr}(A) = \overline{A} \cap \overline{X\backslash A}.$$

Therefore, in a topological space, the frontier of any set is closed, and is equal to the frontier of its complement.

It is noteworthy that an $x \in X$ is a frontier point of a set A if and only if each neighborhood of x meets both A and $X\backslash A$. A frontier point of A can belong to A or not.

Proposition 1.

For any set A in a topological space X, the interior $\mathring{A}$, the frontier $\mathrm{Fr}(A)$, and the exterior $\overset{\circ}{\widehat{X\backslash A}}$, are mutually disjoint and their union is the whole space X.

Proof :

Let a point x of X be in $\mathring{A} \cap \operatorname{Fr}(A)$. Then A is a neighborhood of x. Since $\operatorname{Fr}(A)$ is a subset of $\overline{X\backslash A}$, it follows that $A \cap (X\backslash A) \neq \emptyset$: a contradiction. Thus $\mathring{A} \cap \operatorname{Fr}(A) = \emptyset$.

With a similar argumentation, we see that the existence of a point x in $\operatorname{Fr}(A) \cap \overset{\circ}{\widehat{X\backslash A}}$ would imply $(X\backslash A) \cap A \neq \emptyset$: a contradiction. Hence $\operatorname{Fr}(A) \cap \overset{\circ}{\widehat{X\backslash A}} = \emptyset$.

Moreover, $\mathring{A} \cap \overset{\circ}{\widehat{X\backslash A}} \subset A \cap (X\backslash A) = \emptyset$, and so $\mathring{A} \cap \overset{\circ}{\widehat{X\backslash A}} = \emptyset$.

Further, $X = \mathring{A} \cup \left(X\backslash \mathring{A}\right)$ and $X\backslash \mathring{A} = \overline{X\backslash A} = \overline{X\backslash A} \cap \left[\overline{A} \cap \left(X\backslash \overline{A}\right)\right]$. By the distributive law, we thus get

$$X\backslash \mathring{A} = \left[\overline{X\backslash A} \cap \overline{A}\right] \cup \left[\overline{X\backslash A} \cap \left(X\backslash \overline{A}\right)\right] = \operatorname{Fr}(A) \cup \left[\overline{X\backslash A} \cap \overset{\circ}{\widehat{X\backslash A}}\right].$$

Since $\overset{\circ}{\widehat{X\backslash A}} \subset \overline{X\backslash A}$, it follows that $X\backslash \mathring{A} = \operatorname{Fr}(A) \cup \overset{\circ}{\widehat{X\backslash A}}$. Therefore

$$X = \mathring{A} \cup \operatorname{Fr}(A) \cup \overset{\circ}{\widehat{X\backslash A}}.$$

□

2. Frontier of a closure and frontier of an interior

In this section, we see how the frontier operator interacts with the closure and interior operators.

Proposition 2.

Let A be a subset of a topological space X. Then $\operatorname{Fr}\left(\overline{A}\right) \subset \operatorname{Fr}(A)$ and $\operatorname{Fr}\left(\mathring{A}\right) \subset \operatorname{Fr}(A)$.

Proof :

By definition, $\operatorname{Fr}(A) = \overline{A} \cap \overline{X\backslash A}$. Moreover,

$$\operatorname{Fr}\left(\overline{A}\right) = \overline{\overline{A}} \cap \overline{X\backslash \overline{A}} = \overline{A} \cap \overline{\overset{\circ}{\widehat{X\backslash A}}}$$

and

$$\operatorname{Fr}\left(\mathring{A}\right) = \overline{\mathring{A}} \cap \overline{X\backslash \mathring{A}} = \overline{\mathring{A}} \cap \overline{\overline{X\backslash A}} = \overline{\mathring{A}} \cap \overline{X\backslash A}.$$

Since $\overset{\circ}{\widehat{X\backslash A}} \subset X\backslash A$ and $\mathring{A} \subset A$, it follows that

$$\operatorname{Fr}\left(\overline{A}\right) \subset \overline{A} \cap \overline{X\backslash A} \qquad \text{and} \qquad \operatorname{Fr}\left(\mathring{A}\right) \subset \overline{A} \cap \overline{X\backslash A},$$

that is, $\operatorname{Fr}\left(\overline{A}\right) \subset \operatorname{Fr}(A)$ and $\operatorname{Fr}\left(\mathring{A}\right) \subset \operatorname{Fr}(A)$.

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The sets $\mathrm{Fr}(A)$, $\mathrm{Fr}\left(\overline{A}\right)$ and $\mathrm{Fr}\left(\mathring{A}\right)$ can be pairwise distinct.

For instance, on the real line $X = \mathbb{R}$, for

$$A = \left(\bigcup_{n\in\mathbb{N}}]-n-1,-n[\right) \cup \mathbb{N},$$

these sets are distinct. Indeed, $\overline{A} =]-\infty, 0] \cup \mathbb{N}$, while

$$X\backslash A = (\mathbb{Z}\backslash\mathbb{N}) \cup \left(\bigcup_{n\in\mathbb{N}}]n, n+1[\right)$$

and

$$\overline{X\backslash A} = (\mathbb{Z}\backslash\mathbb{N}) \cup [0,+\infty[.$$

Therefore

$$\mathrm{Fr}(A) = \Big(]-\infty,0] \cup \mathbb{N}\Big) \cap \Big((\mathbb{Z}\backslash\mathbb{N}) \cup [0,+\infty[\Big) = (\mathbb{Z}\backslash\mathbb{N}) \cup \mathbb{N} = \mathbb{Z}.$$

Furthermore,

$$X\backslash\overline{A} = \bigcup_{n\in\mathbb{N}}]n, n+1[\qquad \text{and} \qquad \overline{X\backslash\overline{A}} = [0,+\infty[.$$

It follows that

$$\mathrm{Fr}\left(\overline{A}\right) = \Big(]-\infty,0] \cup \mathbb{N}\Big) \cap [0,+\infty[= \mathbb{N}.$$

In addition,

$$\mathring{A} = \bigcup_{n\in\mathbb{N}}]-n-1,-n[\qquad \text{and} \qquad \overline{\mathring{A}} =]-\infty, 0],$$

while

$$X\backslash\mathring{A} = (\mathbb{Z}\backslash\mathbb{N}) \cup [0,+\infty[\qquad \text{and} \qquad \overline{X\backslash\mathring{A}} = (\mathbb{Z}\backslash\mathbb{N}) \cup [0,+\infty[.$$

This implies

$$\mathrm{Fr}\left(\mathring{A}\right) =]-\infty,0] \cap \Big((\mathbb{Z}\backslash\mathbb{N}) \cup [0,+\infty[\Big) = (\mathbb{Z}\backslash\mathbb{N}) \cup \{0\} = \mathbb{Z}\backslash\mathbb{N}^*.$$

In this special case, we clearly have

$$\mathrm{Fr}\left(\overline{A}\right) \cap \mathrm{Fr}\left(\mathring{A}\right) = \mathbb{N} \cap (\mathbb{Z}\backslash\mathbb{N}^*) = \{0\},$$

while

$$\mathrm{Fr}\left(\overline{A}\right) = \mathbb{N} \subsetneq \mathbb{Z} = \mathrm{Fr}(A)$$

and

$$\mathrm{Fr}\left(\mathring{A}\right) = \mathbb{Z}\backslash\mathbb{N}^* \subsetneq \mathbb{Z} = \mathrm{Fr}(A).$$

3. A property of the frontier of the union of two sets

In this section, we discuss a property of the frontier of the union of two sets.

Proposition 3.

Let A and B be two subsets of a topological space X. Then

$$\mathrm{Fr}(A \cup B) \subset \mathrm{Fr}(A) \cup \mathrm{Fr}(B).$$

Proof :

By definition, $\mathrm{Fr}(A \cup B) = \overline{A \cup B} \cap \overline{X \backslash (A \cup B)}$. Moreover,

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Therefore

$$\mathrm{Fr}(A \cup B) = \left(\overline{A} \cup \overline{B}\right) \cap \overline{X \backslash (A \cup B)}.$$

Since the intersection $\cap$ is distributive over the union $\cup$, it follows that

$$\mathrm{Fr}(A \cup B) = \left[\overline{A} \cap \overline{X \backslash (A \cup B)}\right] \cup \left[\overline{B} \cap \overline{X \backslash (A \cup B)}\right].$$

Further, by the De Morgan's law, we have

$$X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B).$$

This yields $\overline{X \backslash (A \cup B)} \subset \overline{X \backslash A}$ and $\overline{X \backslash (A \cup B)} \subset \overline{X \backslash B}$. Consequently,

$$\mathrm{Fr}(A \cup B) \subset \left(\overline{A} \cap \overline{X \backslash A}\right) \cup \left(\overline{B} \cap \overline{X \backslash B}\right),$$

that is, $\mathrm{Fr}(A \cup B) \subset \mathrm{Fr}(A) \cap \mathrm{Fr}(B)$. □

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> The sets $\mathrm{Fr}(A \cup B)$ and $\mathrm{Fr}(A) \cup \mathrm{Fr}(B)$ can be distinct.

For example, on the real line $X = \mathbb{R}$, for $A = [0, 2]$ and $B = [1, 3]$, the sets $\mathrm{Fr}(A \cup B)$ and $\mathrm{Fr}(A) \cap \mathrm{Fr}(B)$ are distinct. Indeed, for real numbers a and b such that $a < b$, we have

$$\begin{aligned}\mathrm{Fr}\big([a, b]\big) &= \overline{[a, b]} \cap \overline{]-\infty, a[\cup]b, +\infty[} = [a, b] \cap \left(\overline{]-\infty, a[} \cup \overline{]b, +\infty[}\right) \\ &= [a, b] \cap (]-\infty, a] \cup [b, +\infty[) = \big\{a, b\big\};\end{aligned}$$

in particular, $\mathrm{Fr}(A) = \big\{0, 2\big\}$ and $\mathrm{Fr}(B) = \big\{1, 3\big\}$, while

$$\mathrm{Fr}(A \cup B) = \mathrm{Fr}\big([0, 3]\big) = \big\{0, 3\big\} \subsetneq \big\{0, 1, 2, 3\big\} = \mathrm{Fr}(A) \cup \mathrm{Fr}(B).$$

4. A case in which the frontier of a union is the union of the frontiers

The following Proposition 4 allows us to determine a condition such that the frontier of the union of two sets is equal to the union of their frontiers.

Proposition 4.

Let A and B be subsets of a topological space X such that $A \cap B = \emptyset$. Then

$$\Big(\mathrm{Fr}(A) \cup \mathrm{Fr}(B)\Big) \backslash \Big(\big[\mathrm{Fr}(A) \cap B\big] \cup \big[A \cap \mathrm{Fr}(B)\big]\Big) \subset \mathrm{Fr}(A \cup B). \qquad (\dagger)$$

Proof :

Let x be an element of the set

$$C = \Big(\mathrm{Fr}(A) \cup \mathrm{Fr}(B)\Big) \backslash \Big(\big[\mathrm{Fr}(A) \cap B\big] \cup \big[A \cap \mathrm{Fr}(B)\big]\Big).$$

To show that $x \in \mathrm{Fr}(A \cup B)$, we shall distinguish the cases $x \in \mathrm{Fr}(A)$ and $x \in \mathrm{Fr}(B)$.

First case: Let $x \in \mathrm{Fr}(A)$. Then $x \in \overline{A}$, and so $x \in \overline{A \cup B}$. Further, $x \in X \backslash B$; indeed, the contrary would imply $x \in \mathrm{Fr}(A) \cap B$: a contradiction of the hypothesis. Moreover, $x \in X \backslash A$ or $x \in A$.

(i) Let $x \in X \backslash A$. Then $x \in (X \backslash A) \cap (X \backslash B) = X \backslash (A \cup B)$. Therefore

$$x \in \overline{A \cup B} \cap \overline{X \backslash (A \cup B)} = \mathrm{Fr}(A \cup B).$$

(ii) Let $x \in A$. Then $x \notin \mathrm{Fr}(B) = \overline{B} \cap \overline{X \backslash B}$, since $x \notin A \cap \mathrm{Fr}(B)$. Therefore, there is a neighborhood V of x such that

$$V \cap B = \emptyset \qquad \text{or} \qquad V \cap (X \backslash B) = \emptyset.$$

However, the hypothesis $A \cap B = \emptyset$ yields $A \subset X \backslash B$. Thus $x \in V \cap (X \backslash B)$. Hence $V \cap B = \emptyset$, and so $V \subset X \backslash B$. Now let U be a neighborhood of x. Then $U \cap V \cap (X \backslash A) \subset X \backslash B$ and

$$U \cap V \cap (X \backslash A) = U \cap V \cap (X \backslash A) \cap (X \backslash B) \subset U \cap (X \backslash A) \cap (X \backslash B).$$

Furthermore, since $x \in \mathrm{Fr}(A) = \overline{A} \cap \overline{X \backslash A}$, the neighborhood $U \cap V$ of x meets $X \backslash A$. Consequently, $U \cap (X \backslash A) \cap (X \backslash B) \neq \emptyset$ for any neighborhood U of x. Thus

$$x \in \overline{(X \backslash A) \cap (X \backslash B)} = \overline{X \backslash (A \cup B)}.$$

Hence $x \in \mathrm{Fr}(A \cup B)$.

Second case: Let $x \in \mathrm{Fr}(B)$. With an argumentation similar to the one used in the first case, we then obtain $x \in \mathrm{Fr}(A \cup B)$.

In any case, if $x \in C$, then $x \in \mathrm{Fr}(A \cup B)$. This establishes the inclusion $(\dagger)$. $\square$

The following result is a corollary of Proposition 3 and Proposition 4.

Proposition 5.

Let A and B be subsets of a topological space X such that $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$. Then

$$\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B). \qquad (\dagger\dagger)$$

The equality $(\dagger\dagger)$ holds in particular, if $\overline{A} \cap \overline{B} = \emptyset$.

Proof :

The hypothesis implies $A \cap B = \emptyset$, and so the inclusion $(\dagger)$ in Proposition 4. In addition,

$$A \cap \mathrm{Fr}(B) \subset A \cap \overline{B} = \emptyset \qquad \text{and} \qquad \mathrm{Fr}(A) \cap B \subset \overline{A} \cap B = \emptyset.$$

By the inclusion $(\dagger)$, it follows that $\mathrm{Fr}(A) \cup \mathrm{Fr}(B) \subset \mathrm{Fr}(A \cup B)$. Hence

$$\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B),$$

in view of Proposition 3. □

5. A property of the frontier of the intersection of two open sets

In this section, we give a *greatest lower bound* and a *least upper bound* of the frontier of the intersection of two open sets, in the set of subsets of a topological space ordered with inclusion relation.

Proposition 6.

Let A and B be open subsets of a topological space X. Then

$$\Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big) \subset \mathrm{Fr}(A \cap B)$$

and

$$\mathrm{Fr}(A \cap B) \subset \Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big) \cup \Big(\mathrm{Fr}(A) \cap \mathrm{Fr}(B)\Big).$$

Proof :

The definition of the frontier yields

$$A \cap \mathrm{Fr}(B) = A \cap \Big(\overline{B} \cap \overline{X \backslash B}\Big) = \Big(A \cap \overline{B}\Big) \cap \overline{X \backslash B}$$

and

$$B \cap \mathrm{Fr}(A) = B \cap \Big(\overline{A} \cap \overline{X \backslash A}\Big) = \Big(B \cap \overline{A}\Big) \cap \overline{X \backslash A}.$$

However, by Proposition 5, section §1.6 on page 24 in [1], we have

$$A \cap \overline{B} \subset \overline{A \cap B} \qquad \text{and} \qquad B \cap \overline{A} \subset \overline{B \cap A},$$

since A and B are open. Therefore

$$A \cap \mathrm{Fr}(B) \subset \overline{A \cap B} \cap \overline{X \backslash B} \qquad \text{and} \qquad B \cap \mathrm{Fr}(A) \subset \overline{A \cap B} \cap \overline{X \backslash A},$$

and so

$$\Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big) \subset \Big(\overline{A \cap B} \cap \overline{X \backslash B}\Big) \cup \Big(\overline{A \cap B} \cap \overline{X \backslash A}\Big).$$

In addition, from the distributive law, we get

$$\begin{aligned}
\Big(\overline{A \cap B} \cap \overline{X \backslash B}\Big) \cup \Big(\overline{A \cap B} \cap \overline{X \backslash A}\Big) &= \overline{A \cap B} \cap \Big(\overline{X \backslash B} \cup \overline{X \backslash A}\Big) \\
&= \overline{A \cap B} \cap \overline{(X \backslash A) \cup (X \backslash B)} \\
&= \overline{A \cap B} \cap \overline{X \backslash (A \cap B)} \\
&= \mathrm{Fr}(A \cap B).
\end{aligned}$$

Hence,

$$\Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big) \subset \mathrm{Fr}(A \cap B).$$

Furthermore,

$$\mathrm{Fr}(A \cap B) = \overline{A \cap B} \cap \Big(\overline{X \backslash A} \cup \overline{X \backslash B}\Big) \subset \Big(\overline{A} \cap \overline{B}\Big) \cap \Big(\overline{X \backslash A} \cup \overline{X \backslash B}\Big)$$

and

$$\begin{aligned}
\Big(\overline{A} \cap \overline{B}\Big) \cap \Big(\overline{X \backslash A} \cup \overline{X \backslash B}\Big) &= \Big[(\overline{A} \cap \overline{B}) \cap \overline{X \backslash A}\Big] \cup \Big[(\overline{A} \cap \overline{B}) \cap \overline{X \backslash B}\Big] \\
&= \Big[\overline{B} \cap \Big(\overline{A} \cap \overline{X \backslash A}\Big)\Big] \cup \Big[\overline{A} \cap \Big(\overline{B} \cap \overline{X \backslash B}\Big)\Big] \\
&= \big[\overline{B} \cap \mathrm{Fr}(A)\big] \cup \big[\overline{A} \cap \mathrm{Fr}(B)\big].
\end{aligned}$$

Moreover,

$$\overline{B} = \mathring{B} \cup \mathrm{Fr}(B) = B \cup \mathrm{Fr}(B) \qquad \text{and} \qquad \overline{A} = \mathring{A} \cup \mathrm{Fr}(A) = A \cup \mathrm{Fr}(A),$$

since A and B are open. It follows that

$$\begin{aligned}
\Big(\overline{A} \cap \overline{B}\Big) \cap \Big(\overline{X \backslash A} \cup \overline{X \backslash B}\Big) &= \Big[\Big(B \cup \mathrm{Fr}(B)\Big) \cap \mathrm{Fr}(A)\Big] \cup \\
&\qquad \Big[\Big(A \cup \mathrm{Fr}(A)\Big) \cap \mathrm{Fr}(B)\Big] \\
&= \Big(B \cap \mathrm{Fr}(A)\Big) \cup \Big(\mathrm{Fr}(B) \cap \mathrm{Fr}(A)\Big) \cup \\
&\qquad \Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(\mathrm{Fr}(A) \cap \mathrm{Fr}(B)\Big).
\end{aligned}$$

Consequently,

$$\mathrm{Fr}(A \cap B) \subset \Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big) \cup \Big(\mathrm{Fr}(A) \cap \mathrm{Fr}(B)\Big).$$

□

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The three sets $\Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big)$ and $\mathrm{Fr}(A \cap B)$, plus

$$\Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big) \cup \Big(\mathrm{Fr}(A) \cap \mathrm{Fr}(B)\Big)$$

can be distincts.

For instance, on the real line $X = \mathbb{R}$, for $A =]0,1[\cup]3,4[$ and $B =]1,2[\cup]3,5[$, these three sets are distinct. Indeed, $A \cap B =]3,4[$ and

$$\mathrm{Fr}(A \cap B) = \Big\{3,4\Big\}.$$

Furthermore,

$$\overline{]0,1[} \cap \overline{]3,4[} = [0,1] \cap [3,4] = \emptyset \qquad \text{and} \qquad \overline{]1,2[} \cap \overline{]3,5[} = [1,2] \cap [3,5] = \emptyset.$$

By Proposition 5, it follows that

$$\mathrm{Fr}(A) = \mathrm{Fr}\Big(]0,1[\Big) \cup \mathrm{Fr}\Big(]3,4[\Big) = \Big\{0,1\Big\} \cup \Big\{3,4\Big\} = \Big\{0,1,3,4\Big\}$$

and

$$\mathrm{Fr}(B) = \mathrm{Fr}\Big(]1,2[\Big) \cup \mathrm{Fr}\Big(]3,5[\Big) = \Big\{1,2\Big\} \cup \Big\{3,5\Big\} = \Big\{1,2,3,5\Big\}.$$

Therefore

$$A \cap \mathrm{Fr}(B) = \emptyset \qquad \text{and} \qquad B \cap \mathrm{Fr}(A) = \Big\{4\Big\},$$

while

$$\mathrm{Fr}(A) \cap \mathrm{Fr}(B) = \Big\{1,3\Big\}.$$

Thus

$$\underbrace{\Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big)}_{\{4\}} \subsetneq \underbrace{\mathrm{Fr}(A \cap B)}_{\{3,4\}}$$

and

$$\underbrace{\mathrm{Fr}(A \cap B)}_{\{3,4\}} \subsetneq \underbrace{\Big(A \cap \mathrm{Fr}(B)\Big) \cup \Big(B \cap \mathrm{Fr}(A)\Big) \cup \Big(\mathrm{Fr}(A) \cap \mathrm{Fr}(B)\Big)}_{\{1,3,4\}}.$$

References

[1] Bourbaki, N., *Elements of mathematics*, **General Topology**, Chapters 1 - 4, Springer-Verlag, Berlin, etc., 1989.

[2] Kelley, J. L., *General Topology*, Graduate Texts in Mathematics **27**, 2nd printing, Springer-Verlag, New York, etc., 1975.