The right topology, the left topology and Kolmogoroff spaces

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Given an ordered set, several topologies can be defined on the underlying set by means of the ordering. In this paper, we introduce two of these topologies: the *right topology* and the *left topology*. We also give an introduction to *Kolmogoroff spaces*. The text is in fact an organized composition on tasks assigned in Exercise 2 of the section §1 in Chapter I of the volume on *General Topology* by Bourbaki [1].

The paper is divided into four sections. The first section gives the respective definitions of the right topology and the left topology, as well as some of their features. The second section is devoted to the presentation of the Kolmogoroff spaces. We shall see that the left and right topologies form an important source for Kolmogoroff spaces. In the third section, we show that an ordering can be defined on any Kolmogoroff space, and that the given topology on the Kolmogoroff space is identical with the right topology determined by this ordering, in some cases. The fourth section discuss nature of isolated points in Kolmogoroff spaces. In particular, we will establish a noteworthy consequence of the lack of isolated points in a Kolmogoroff space.

Before getting to the heart of the matter, we recall that a set $\mathfrak{B}$ of subsets of a topological space X is said to be a **base** of X (or a **base** of the topology on X) if the open subsets of X are the unions of sets belonging to $\mathfrak{B}$.

We also recall without proof two useful results on the bases of topology.

Proposition 1.

Let $\mathfrak{B}$ be a set of subsets of a topological space X. Then the following statements are equivalent:

(1) $\mathfrak{B}$ is a base of the topology on X.

(2) For each $x \in X$, the set of $B \in \mathfrak{B}$ such that $x \in B$ is a fundamental system of neighborhoods of x.

(3) A set U is an open subset of X if and only if, for any $x \in U$, there is a set $B \in \mathfrak{B}$ such that $x \in B \subset U$.

Proposition 2.

Let X be a set. A subset $\mathfrak{B}$ of $\mathscr{P}(X)$ is a base of a topology on X if and only if the following conditions are satisfied:

(1) X is the union of $\mathfrak{B}$.

(2) For each pair (A, B) of sets of $\mathfrak{B}$, if $x \in A \cap B$, then there exists a $C \in \mathfrak{B}$ such that $x \in C \subset A \cap B$.

1. The right topology and the left topology

In an ordered set X, the *closed interval unbounded on the right* with *left-hand endpoint* $x \in X$ is the set denoted by $[x, \rightarrow[$ and defined by

$$[x, \rightarrow[\, = \Big\{ y \in X \mid x \leqslant y \Big\}.$$

The *closed interval unbounded on the left* with *right-hand endpoint* $x \in X$ is the set denoted by $]\leftarrow, x]$ and defined by

$$]\leftarrow, x] = \Big\{ y \in X \mid y \leqslant x \Big\}.$$

1.1. The right topology

Proposition 3.

Let X be an ordered set. Then the set of intervals $[x, \rightarrow[$, where x runs through X, is a base of a topology on X; this topology is called the **right topology** on X.

Proof :

Let $\mathfrak{B} = \Big\{ [x, \rightarrow[\mid x \in X \Big\}$. By the definition of intervals, the inclusions $\{x\} \subset [x, \rightarrow[$ and $[x, \rightarrow[\subset X$ hold for each $x \in X$. This implies

$$X = \bigcup \Big\{ \{x\} \mid x \in X \Big\} \subset \bigcup \Big\{ [x, \rightarrow[\mid x \in X \Big\} = \bigcup \mathfrak{B}$$

and

$$\bigcup \mathfrak{B} = \bigcup \Big\{ [x, \rightarrow[\mid x \in X \Big\} \subset X.$$

Therefore $X = \bigcup \mathfrak{B}$. We now consider two elements a and b of X. If $x \in [a, \rightarrow[\cap [b, \rightarrow[$ and $y \in [x, \rightarrow[$, then $a \leqslant x$ and $b \leqslant x$, while $x \leqslant y$. By the transitivity of the order relation on X, this yields $a \leqslant y$ and $b \leqslant y$. Thus $[x, \rightarrow[\subset [a, \rightarrow[\cap [b, \rightarrow[$. We thus have established that, for every pair $(A, B) \in \mathfrak{B} \times \mathfrak{B}$, if $x \in A \cap B$, then there exists a $C \in \mathfrak{B}$ such that $x \in C \subset A \cap B$. Hence $\mathfrak{B}$ is a base of a topology on X, by Proposition 2. $\square$

The following Proposition 4 reveals two noteworthy features of the right topology.

Proposition 4.

In the right topology on an ordered set X, any intersection of open sets is an open set, and the closure of any singleton $\{x\}$ is the interval $]\leftarrow, x]$.

Proof :

Let $\mathfrak{O}$ be the set of open subsets of the ordered set X endowed with the right topology. We also consider a subset $\mathfrak{A}$ of $\mathfrak{O}$. Let x be a point of $\bigcap \mathfrak{A}$. Then $x \in A$ for each $A \in \mathfrak{A}$, and so there is a point a in X such that $x \in [a, \rightarrow[\subset A$. Moreover, $x \in [a, \rightarrow[$ implies $[x, \rightarrow[\subset [a, \rightarrow[$, by the transitivity of the order relation on X. Therefore $[x, \rightarrow[\subset A$ for each $A \in \mathfrak{A}$, and so $[x, \rightarrow[\subset \bigcap \mathfrak{A}$. Hence $\bigcap \mathfrak{A}$ is a neighborhood of each of its points, that is, $\bigcap \mathfrak{A} \in \mathfrak{O}$.

Let x be a point of X. We first consider a point y of the closure of $\{x\}$. Then each neighborhood of y meets $\{x\}$; in particular, $[y, \rightarrow[\cap \{x\} \neq \emptyset$. Therefore $x \in [y, \rightarrow[$, that is, $y \leqslant x$. Thus $y \in]\leftarrow, x]$. Hence the closure of $\{x\}$ is contained in the interval $]\leftarrow, x]$. Conversely, we consider a point $z \in]\leftarrow, x]$ and a neighborhood V of z. Then there exists an $a \in X$ such that $z \in [a, \rightarrow[\subset V$. Therefore $[z, \rightarrow[\subset V$. Since $z \leqslant x$, it follows that $x \in V$ and $\{x\} \cap V = \{x\}$. Thus every neighborhood of z meets $\{x\}$, and so any point z of the interval $]\leftarrow, x]$ is in the closure of $\{x\}$. Consequently, $\overline{\{x\}} =]\leftarrow, x]$. $\square$

1.2. The left topology

Proposition 5.

Let X be an ordered set. Then the set of intervals $]\leftarrow, x]$, where x runs through X, is a base of a topology on X; this topology is called the **left topology** on X.

Proof :

Let $\mathfrak{B} = \Big\{]\leftarrow, x] \mid x \in X \Big\}$. By the definition of intervals, the inclusions $\{x\} \subset]\leftarrow, x]$ and $]\leftarrow, x] \subset X$ hold for each $x \in X$. This implies

$$X = \bigcup \Big\{ \{x\} \mid x \in X \Big\} \subset \bigcup \Big\{]\leftarrow, x] \mid x \in X \Big\} = \bigcup \mathfrak{B}$$

and

$$\bigcup \mathfrak{B} = \bigcup \Big\{]\leftarrow, x] \mid x \in X \Big\} \subset X.$$

Therefore $X = \bigcup \mathfrak{B}$. We now consider two elements a and b of X. If $x \in]\leftarrow, a] \cap]\leftarrow, b]$ and $y \in]\leftarrow, x]$, then $y \leqslant x$, while $x \leqslant a$ and $x \leqslant b$. By the transitivity of the order relation on X, this yields $y \leqslant a$ and $y \leqslant b$. Thus $]\leftarrow, x] \subset]\leftarrow, a] \cap]\leftarrow, b]$. We thus have established that, for every pair $(A, B) \in \mathfrak{B} \times \mathfrak{B}$, if $x \in A \cap B$, then there exists a $C \in \mathfrak{B}$ such that $x \in C \subset A \cap B$. Hence $\mathfrak{B}$ is a base of a topology on X. $\square$

The following Proposition 6 gives two noteworthy features of the left topology.

Proposition 6.

In the left topology on an ordered set X, any intersection of open sets is an open set, and the closure of any singleton $\{x\}$ is the interval $[x, \rightarrow [$.

Proof :

Let $\mathfrak{O}$ be the set of open subsets of the ordered set X endowed with the left topology. We also consider a subset $\mathfrak{A}$ of $\mathfrak{O}$. Let x be a point of $\bigcap \mathfrak{A}$. Then $x \in A$ for each $A \in \mathfrak{A}$, and so there is a point a in X such that $x \in] \leftarrow, a] \subset A$. Moreover, $x \in] \leftarrow, a]$ implies $] \leftarrow, x] \subset] \leftarrow, a]$, by the transitivity of the order relation on X. Therefore $] \leftarrow, x] \subset A$ for each $A \in \mathfrak{A}$, and so $] \leftarrow, x] \subset \bigcap \mathfrak{A}$. Hence $\bigcap \mathfrak{A}$ is a neighborhood of each of its points, that is, $\bigcap \mathfrak{A} \in \mathfrak{O}$.

Let x be a point of X. We first consider a point y of the closure of $\{x\}$. Then each neighborhood of y meets $\{x\}$; in particular, $] \leftarrow, y] \cap \{x\} \neq \emptyset$. Therefore $x \in] \leftarrow, y]$, that is, $x \leqslant y$. Thus $y \in [x, \rightarrow [$. Hence the closure of $\{x\}$ is contained in the interval $[x, \rightarrow [$. Conversely, we consider a point $z \in [x, \rightarrow [$ and a neighborhood V of z. Then there exists an $a \in X$ such that $z \in] \leftarrow, a] \subset V$. Therefore $] \leftarrow, z] \subset V$. Since $x \leqslant z$, it follows that $x \in V$ and $\{x\} \cap V = \{x\}$. Thus, every neighborhood of z meets $\{x\}$, and so any point z of the interval $[x, \rightarrow [$ is in the closure of $\{x\}$. Consequently, $\overline{\{x\}} = [x, \rightarrow [$. □

2. Kolmogoroff spaces

Definition 1.

A topological space X is said to be a **Kolmogoroff space** if it satisfies the following condition: given any two district points x and x' of X, there is a neighborhood of one of these points which does not contain the other.

Two classes of Kolmogoroff spaces are given by the Propositions 7 and 8 below.

Proposition 7.

An ordered set with the right topology is a Kolmogoroff space.

Proof :

Let x et x' be two distinct points of an ordered set X endowed with the right topology. We assume that each neighborhood of x contains x' and each neighborhood of x' contains x. This means $x \in \overline{\{x'\}}$ and $x' \in \overline{\{x\}}$. By Proposition 4 on page 3, it follows that $x \in] \leftarrow, x']$ and $x' \in] \leftarrow, x]$, that is, $x \leqslant x'$ and $x' \leqslant x$. Thus $x = x'$: a contradiction of the hypothesis. The assumption is therefore false. Hence, there is a neighborhood of x which does not contain x' or a neighborhood of x' which does not contain x. Consequently, the ordered set X endowed with the right topology is a Kolmogoroff space. □

Proposition 8.

An ordered set with the left topology is a Kolmogoroff space.

Proof :

Let x et x' be two distinct points of an ordered set X endowed with the left topology. We suppose that each neighborhood of x contains x' and each neighborhood of x' contains x. This implies $x \in \overline{\{x'\}}$ and $x' \in \overline{\{x\}}$. By Proposition 6 on page 4, it follows that $x \in [x', \rightarrow [$ and $x' \in [x, \rightarrow [$, that is, $x' \leqslant x$ and $x \leqslant x'$. Thus $x' = x$: a contradiction of the hypothesis. The assumption is therefore false. Hence, there is a neighborhood of x which does not contain x' or a neighborhood of x' which does not contain x. Consequently, the ordered set X endowed with the left topology is a Kolmogoroff space. □

3. An order relation defined by a Kolmogoroff space

Kolmogoroff spaces can be regarded as topological spaces with a "weak" *separation axiom.* This separation axiom suggests a binary relation on the underlying set. We will show that this binary relation is in fact an order relation, which, in a particular case, can determine the given topology on the Kolmogoroff space.

Proposition 9.

Let X be a Kolmogoroff space. Then, an order relation $\leqslant$ is defined on X by $x \leqslant x'$ if $x \in \overline{\{x'\}}$. Moreover, if, in the topological space X, every intersection of open sets is an open set, then the given topology on X is identical with the right topology determined by the ordering $\leqslant$.

Proof :

Reflexivity: Since each subset is contained in its closure, for every $x \in X$, we have $x \in \{x\} \subset \overline{\{x\}}$, and so $x \leqslant x$.

Anti-symmetry: Let x and x' be points of X such that $x \leqslant x'$ and $x' \leqslant x$, that is, $x \in \overline{\{x'\}}$ and $x' \in \overline{\{x\}}$. Then every neighborhood of x contains x' and every neighborhood of x' contains x. It follows that $x = x'$, since X is a Kolmogoroff space.

Transitivity: Let x, y and z be points of X such that $x \leqslant y$ and $y \leqslant z$, that is, $x \in \overline{\{y\}}$ and $y \in \overline{\{z\}}$. Then every neighborhood of x contains y and every neighborhood of y contains z. Now we consider a neighborhood V of x. Then there is an open neighborhood U of x such that $x \in U \subset V$. Thus $y \in U$, and so U is a neighborhood of y. It follows that $z \in U \subset V$. Hence z is contained in every neighborhood of x. This implies $x \in \overline{\{z\}}$, that is, $x \leqslant z$.

The pair $(X, \leqslant)$ is thus an ordered set.

We now assume that, in the topological space X, every intersection of open sets is an open set.

Let U be an open subset of the Kolmogoroff space X and $x \in U$. If $y \in [x, \rightarrow [$, that is, $x \in \overline{\{y\}}$, then every neighborhood of x contains y; in particular, $y \in U$. Therefore $x \in [x, \rightarrow [\subset U$ for any $x \in U$. Thus

$$U = \bigcup \Big\{ [x, \rightarrow [\ \Big| \ x \in U \Big\}.$$

Hence each open subset of the Kolmogoroff space X is a union of closed intervals, unbounded on the right, of the ordered set $(X, \leqslant)$.

We now consider a point $x \in X$ and the set $\mathfrak{A}$ of all open neighborhood of x. If $y \in \bigcap \mathfrak{A}$, then every neighborhood of x contains y, since $\mathfrak{A}$ is a fundamental system of neighborhoods of x; and so $x \in \overline{\{y\}}$, that is, $x \leqslant y$. Conversely, if $y \in [x, \rightarrow [$, then y belongs to each neighborhood of x; in particular $y \in \bigcap \mathfrak{A}$. Hence $[x, \rightarrow [= \bigcap \mathfrak{A}$. It follows that each interval $[x, \rightarrow [$, where $x \in X$, is an open subset of the Kolmogoroff space X, since every intersection of open sets is an open set.

Consequently, the given topology on X is identical with the right topology determined by the ordering $\leqslant$. $\square$

4. A consequence of the lack of isolated points in a Kolmogoroff space

Definition 2.

Let X be a topological space and A a subset of X. A point $x \in A$ is said to be an **isolated point** of A if x has a neighborhood which contains no point of A except x; this is equivalent to saying that there exists an open subset U of X such that $U \cap A = \{x\}$, that is, the singleton $\{x\}$ is an open subset of the subspace A.

Clearly, any point x of X is an isolated point of the singleton $\{x\}$. Moreover, a point x is isolated in the whole space X if and only if the singleton $\{x\}$ is an open subset of X.

In Kolmogoroff spaces, isolated points of finite subsets can be characterized by means of the order relation introduced in Proposition 9 above.

Proposition 10.

Let X be a Kolmogoroff space. Then, an element of a non-empty finite subset F is an **isolated point** of F if and only if it is a **maximal element** of F with respect to the order relation $\leqslant$ defined on the underlying set of the Kolmogoroff space X.

Proof :

By the definition of Kolmogoroff spaces, given any two distinct points x and x' of X, we have $x \notin \overline{\{x'\}}$ or $x' \notin \overline{\{x\}}$. Moreover, by Proposition 9, an order relation $\leqslant$ is defined by $x \leqslant x'$ if $x \in \overline{\{x'\}}$. Therefore, given any two distinct points x and x' of X, we have $x \nleqslant x'$ or $x' \nleqslant x$, that is, the pair $\{x, x'\}$ has a maximal element.

Now, let F be a non-empty finite subset of X. By Definition 2, an $a \in F$ is an isolated point of F if and only if there is a neighborhood V of a such that $V \cap \big(F\backslash\{a\}\big) = \emptyset$, that is, $a \notin \overline{F\backslash\{a\}}$. Since

$$\overline{F\backslash\{a\}} = \overline{\bigcup_{b \in F\backslash\{a\}} \{b\}} = \bigcup_{b \in F\backslash\{a\}} \overline{\{b\}},$$

this is equivalent to $a \notin \overline{\{b\}}$ for each $b \in F\backslash\{a\}$. In other words, $a \nleqslant b$ for each $b \in F\backslash\{a\}$. This means that the element a is maximal in F. □

So, to prove that a subset of a Kolmogoroff space has an isolated point, it suffices to show that it possesses a maximal element with respect to the order relation defined in Proposition 9 above. We use this principle in the proof of the following result.

Proposition 11.

If X is a Kolmogoroff space, then every finite non-empty subset of X has at least one isolated point.

Proof :

Let X be a Kolmogoroff space. Clearly, each singleton contained in X has a maximal element. Also, we saw in the proof of Proposition 10 that any pair of distinct points of X has a maximal element.

We now consider an integer $n \geqslant 1$ and assume that every non-empty subset of X, with a cardinal lower or equal to n, has a maximal element. Let F be subset of X such that $\operatorname{card}(F) = n+1$. We pick a $b \in F$. Then, by the assumption, the subset $F\backslash\{b\}$ has a maximal element a. Thus $a \nleqslant x$ for each $x \in F\backslash\{a\}$. Further, since $a \neq b$, we have $a \nleqslant b$ or $b \nleqslant a$.

First case: Let $a \nleqslant b$. Then $a \nleqslant x$ for each $x \in F$; and so a is a maximal element of F.

Second case: Let $b \nleqslant a$ and set

$$E = [b, \rightarrow [\, \cap F = \Big\{x \in F \mid b \leqslant x\Big\}.$$

Then $a \notin E$, and so $\operatorname{card}(E) \leqslant n$. By the assumption, it follows that E has a maximal element c. We now suppose that there is an $x \in F\backslash\{c\}$ such that $c \leqslant x$. Then, from $b \leqslant c$, we get by transitivity $b \leqslant x$, that is $x \in E$; this contradicts the maximality of c in E. The supposition is thus false. In other words, $c \nleqslant x$ for each $x \in F\backslash\{c\}$. Hence c is a maximal element of F.

We so have shown by induction that every non-empty finite subset of X has at least one maximal element, that is, at least one isolated point. □

Alternate Proof (without the use of the order relation) :

We shall prove by induction on the cardinal of F the following property $(\mathbf{P})$: for every non-empty finite subset F of X, there are a point $a \in F$ and a neighborhood V of a such that $V \cap F = \{a\}$.

The property $(\mathbf{P})$ is certainly true if $\text{card}(F) = 1$; indeed, for each point a in X and any neighborhood of a, we have $V \cap \{a\} = \{a\}$.

We now assume that the property $(\mathbf{P})$ is true for each non-empty finite subset of X whose cardinal does not exceed an integer $n \geqslant 1$. Let F be a non-empty finite subset of X such that $\text{card}(F) = n + 1$. We take a point $b \in F$. Then, by the induction hypothesis, there is a point $a \in F\backslash\{b\}$ and a neighborhood U of a such that $U \cap \big(F\backslash\{b\}\big) = \{a\}$. Moreover, since $a \neq b$, there exists a neighborhood V_a of a such $b \notin V_a$ or an open neighborhood V_b of b such that $a \notin V_b$.

First case: There exists a neighborhood V_a of a such $b \notin V_a$. Then $U \cap V_a$ is a neighborhood of a, and

$$\begin{aligned}
(U \cap V_a) \cap F &= \big(U \cap V_a\big) \cap \Big[\big(F\backslash\{b\}\big) \cup \{b\}\Big] \\
&= \Big[\big(U \cap V_a\big) \cap \big(F\backslash\{b\}\big)\Big] \cup \Big[\big(U \cap V_a\big) \cap \{b\}\Big] \\
&= \Big[\Big(U \cap \big(F\backslash\{b\}\big)\Big) \cap V_a\Big] \cup \Big[U \cap \Big(V_a \cap \{b\}\Big)\Big] \\
&= \Big[\{a\} \cap V_a\Big] \cup \Big[U \cap \emptyset\Big] = \{a\} \cup \emptyset = \{a\}.
\end{aligned}$$

Second case: There exists an open neighborhood V_b of b such $a \notin V_b$. Then, we have $b \in V_b \cap F \subset F\backslash\{a\}$, and so $1 \leqslant \text{card}(V_b \cap F) \leqslant n$. Therefore, by the induction hypothesis, there is a point $c \in V_b \cap F$ and an neighborhood W of c such that $W \cap (V_b \cap F) = \{c\}$, that is $(W \cap V_b) \cap F = \{c\}$, where $W \cap V_b$ is a neighborhood of c, since V_b is an open subset of X containing c.

We so have shown by induction that, for every non-empty finite subset F of X, there are a point $a \in F$ and a neighborhood V of a such that $V \cap F = \{a\}$. In other words, every finite non-empty subset of X has at least one isolated point. □

We are now equipped to prove a consequence of the lack of isolated points in a Kolmogoroff space, as announced in title of the section.

Proposition 12.

If a Kolmogoroff space X has no isolated point, every non-empty open subset of X is infinite.

Proof :

Let X be a Kolmogoroff space with no isolated point. We assume that there is a non-empty open subset F which is finite. Then, by Proposition 11 above, F has an isolated point a. Therefore, there is an open neighborhood U of a such that $U \cap F = \{a\}$, and so the singleton $\{a\}$ is an open subset of X, that is, a is an isolated point of X; this contradicts the hypothesis. The assumption is thus false. Hence, every non-empty open subset of X is infinite. $\square$

References

[1] Bourbaki, N., *Elements of mathematics*, **General Topology**, Chapters 1 - 4, Springer-Verlag, Berlin, etc., 1989.

[2] Kelley, J. L., *General Topology*, Graduate Texts in Mathematics **27**, 2nd printing, Springer-Verlag, New York, etc., 1975.