The Kuratowski Closure-Complement Problem

1. The Kuratowski's Monoid

We recall that a **magma** is a set M endowed with a composition law

$$M \times M \to M, (x, y) \mapsto xy.$$

The composition of a finite sequence $(x_j)_{1 \leqslant j \leqslant n}$, where n is a non-zero natural number, is defined by induction as follow:

- if $n = 1$, then $\prod_{j=1}^{n} x_j = x_1$;
- if $n \geqslant 2$, then $\prod_{j=1}^{n} x_j = x_1 \left(\prod_{j=2}^{n} x_j \right)$.

In particular, the **n-th power** of an element x of M is defined inductively by

$$x^1 = x \qquad \text{and} \qquad x^n = x\left(x^{n-1}\right)$$

if $n \geqslant 2$. Moreover, if the magma M has an identity e, we define $x^0 = e$ for any element x of X; we also set that the composition of every empty sequence to be equal to the identity e; in other words, $\prod_{j \in \emptyset} x_j = e$.

(1) Let M be an *associative* magma and S a subset of M. Show that each element of the **stable subset of** M **generated by** S, that is, the *smallest* stable subset of M containing S, has the form $x_1 x_2 \cdots x_n$, where n is a non-zero natural number, and the x_1, x_2, …, x_n are elements of S.

(2) We recall that a **monoid** is an associative magma with an identity. Let M be a monoid and e its identity. Prove that any element of the *sub-monoid generated* by a subset S of M is a finite composition of powers of elements of S, that is, the composition of an empty sequence or an element of the form

$$\prod_{j=1}^{k} x_j^{n_j} = x_1^{n_1} \cdots x_k^{n_k},$$

where k is a non-zero natural number, the n_j are natural numbers, and the x_j are elements of S. In particular, the sub-monoid of M generated by the empty set is the singleton $\{e\}$.

(3) Let M be a monoid and e its identity. Let also i and c be two distinct elements of $M\backslash\{e\}$ such that $i^2 = i$ and $c^2 = c$, as well as $(ic)^2 = ic$ and $(ci)^2 = ci$. Show that each element of the sub-monoid of M generated by i and c is equal to one of the following compositions:

$$e, \quad i, \quad c, \quad ic, \quad ci, \quad ici, \quad cic.$$

(4) Let M be a monoid and e its identity. Let also d and c be two distinct elements of $M\backslash\{e\}$ such that $d^2 = e$ and $c^2 = c$, as well as $(dcdc)^2 = dcdc$. Show that each element of the sub-monoid of M generated by d and c is equal to one of the following compositions:

$$e, \quad d, \quad c, \quad dc, \quad cd, \quad dcd, \quad cdc, \quad dcdc, \quad cdcd, \quad dcdcd, \quad cdcdc,$$
$$dcdcdc, \quad cdcdcd, \quad dcdcdcd.$$

2. The Kuratowski Closure-Complement Theorem

The goal of this part is to prove a theorem introduced in 1922 by the polish mathematician Kazimierz Kuratowski.

(1) Let X be a topological space. For any subset A of X, let $\alpha(A)$ denote $\mathring{\overline{A}}$, the interior of the closure of A, and $\beta(A)$ denote $\overline{\mathring{A}}$, the closure of the interior of A. Clearly, if A and B are subsets of X such that $A \subset B$, then $\alpha(A) \subset \alpha(B)$ and $\beta(A) \subset \beta(B)$.

- **(a)** Show that, if A is an open subset of X, then $A \subset \alpha(A)$.
- **(b)** Show that, if A is a closed subset of X, then $\beta(A) \subset A$.
- **(c)** Deduce from **(a)** and **(b)** that $\alpha\big(\alpha(A)\big) = \alpha(A)$ for each subset A of X.
- **(d)** Deduce from **(a)** and **(b)** that $\beta\big(\beta(A)\big) = \beta(A)$ for each subset A of X.

(2) Prove that, from a given subset A of a topological space X and counting A itself, *at most* seven sets can be constructed by applying interior and closure successively; namely, the sets

$$A, \quad \mathring{A}, \quad \overline{A}, \quad \alpha(A), \quad \beta(A), \quad \alpha\left(\mathring{A}\right) \quad \text{and} \quad \beta\left(\overline{A}\right).$$

Show also that, on the real line (the set $\mathbb{R}$ equipped with its natural topology), there is a subset from which seven *different* sets can be so constructed.

(3) **The Kuratowski Closure-Complement Theorem:**

From a given subset A of a topological space X and counting A itself, *at most* 14 sets can be constructed by applying complementation and closure successively. Further, on the real line, there is a subset from which 14 *different* sets can be so constructed.

Prove the preceding theorem.

(4) From a given subset A of a topological space X and counting A itself, how many sets can be constructed by applying complementation, closure and interior successively?