Topologies from closure operators

Christian V. Nguembou Tagne

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In this note, we prove the following result.

Proposition 1.

Let X be a set and let $M \mapsto \overline{M}$ be a mapping of the power set $\mathscr{P}(X)$ onto itself such that:

(1) $\overline{\emptyset} = \emptyset$;

(2) for every $M \subset X$, we have $M \subset \overline{M}$;

(3) for every $M \subset X$, we have $\overline{\overline{M}} = \overline{M}$;

(4) for all $M \subset X$ and $N \subset X$, we have $\overline{M \cup N} = \overline{M} \cup \overline{N}$.

Then there is a unique topology on X such that $\overline{M}$ is the closure of M with respect to this topology, for all $M \subset X$.

Proof :

There are two parts to our argument.

Existence of the topology

Let $\mathfrak{F}$ the set of all subsets A of X satisfying $\overline{A} = A$. Then $\emptyset \in \mathfrak{F}$, by **(1)**. Moreover, by **(4)**, if A and B are members of $\mathfrak{F}$, then

$$\overline{A \cup B} = \overline{A} \cup \overline{B} = A \cup B,$$

and so $A \cup B$ is also a member of $\mathfrak{F}$. Therefore, every *finite union* of sets of $\mathfrak{F}$ is a set of $\mathfrak{F}$. Further, $\overline{X} \subset X$, by **(2)**. However, by definition, $\overline{X} \subset X$. Therefore $\overline{X} = X$, and so $X \in \mathfrak{F}$. We shall now prove that every non-empty intersection of sets of $\mathfrak{F}$ is a set of $\mathfrak{F}$. To this end, we note that, if $B \subset A$, then $\overline{B} \subset \overline{A}$; indeed, $B \subset A$ implies $A = (A \backslash B) \cup B$ and

$$\overline{A} = \overline{(A \backslash B) \cup B} = \overline{(A \backslash B)} \cup \overline{B},$$

and so $\overline{B} \subset \overline{A}$. Let B be the intersection of a non-empty set $\mathfrak{A}$ of members of $\mathfrak{F}$. Then, for each $A \in \mathfrak{A}$, we have $B \subset A$, and so $\overline{B} \subset \overline{A} = A$. Therefore $\overline{B} \subset \bigcap \mathfrak{A} = B$. In addition, $B \subset \overline{B}$, by the property **(2)**. Thus $\overline{B} = B$, that is, $B \in \mathfrak{F}$. It follows that every *intersection* of sets of $\mathfrak{F}$ is a set of $\mathfrak{F}$. Hence $\mathfrak{F}$ is the set of closed sets of a topology $\mathfrak{T}$ on X.

Closure of sets and uniqueness of the topology

For any subset A of X, let A^c denote the closure of A with respect to the topology $\mathfrak{T}$. Then $A^c = \bigcap \mathfrak{B}$, where $\mathfrak{B}$ is the set of all closed sets, which contain A. In other words, $\mathfrak{B}$ is the set of all members of $\mathfrak{F}$, which contain A. The property **(3)** implies $\overline{A} \in \mathfrak{F}$, while the property **(2)** yields $A \subset \overline{A}$. Therefore $\overline{A}$ is a member of $\mathfrak{B}$. Hence $A^c \subset \overline{A}$. Further, we get $\overline{A} \subset \overline{A^c}$ from $A \subset A^c$. However, A^c belongs to $\mathfrak{F}$, as intersection of members of $\mathfrak{F}$. Thus $\overline{A^c} = A^c$, and so $\overline{A} \subset A^c$. Consequently, $A^c = \overline{A}$. From this equality, since the closed sets of a topological space are the ones coinciding with their respective closures, we deduce the uniqueness of the topology $\mathfrak{T}$. □

References

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