The characteristic function and the Sierpinski-space

Christian V. Nguembou Tagne

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In this note, we give a necessary and sufficient condition for the continuity of a characteristic function from a topological space into the Sierpinski-space.

1. The characteristic function

We set $\mathbf{2} = \{0, 1\}$ and recall that the **characteristic function** of a subset A of a set X is the function denoted by χ_A, from X into $\mathbf{2}$, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

> To prove that the function $\gamma : \mathscr{P}(X) \to \mathbf{2}^X$, given by $\gamma(A) = \chi_A$, is a bijection, we will exhibit its inverse function.

Proof :

A function $\delta : \mathbf{2}^X \to \mathscr{P}(X)$ is defined by $\delta(f) = f^{-1}(\{1\})$, where

$$f^{-1}(\{1\}) = \{x \in X \ : \ f(x) = 1\}$$

is the inverse image of the subset $\{1\}$ by f. We shall show that δ is the inverse image of γ, that is, $\gamma \circ \delta = \mathrm{id}_{\mathbf{2}^X}$ et $\delta \circ \gamma = \mathrm{id}_{\mathscr{P}(X)}$.

Firstly, let f be a function from X into $\mathbf{2}$. Then $(\gamma \circ \delta)(f) = \gamma\big(f^{-1}(\{1\})\big) = \chi_{f^{-1}(\{1\})}$. However,

$$\chi_{f^{-1}(\{1\})}(x) = \left\{ \begin{array}{l} 1 \text{ if } x \in f^{-1}(\{1\}), \\ 0 \text{ if } x \in X \setminus f^{-1}(\{1\}), \end{array} \right\} = \left\{ \begin{array}{l} 1 \text{ if } f(x) = 1, \\ 0 \text{ if } f(x) = 0, \end{array} \right\} = f(x)$$

for each $x \in X$. Therefore $(\gamma \circ \delta)(f) = f$ for all $f \in \mathbf{2}^X$, that is, $\gamma \circ \delta = \mathrm{id}_{\mathbf{2}^X}$.

Secondly, let A be a subset of X. Then $(\delta \circ \gamma)(A) = \delta(\chi_A) = \chi_A^{-1}(\{1\}) = A$. Thus $(\delta \circ \gamma)(A) = A$ for each $A \in \mathscr{P}(X)$, that is, $\delta \circ \gamma = \mathrm{id}_{\mathscr{P}(X)}$. □

2. The Sierpinski-space

Let X be a set, and A a subset of X. It is easily shown that any intersection of members of $\mathfrak{T} = \{\emptyset, A, X\}$ is always a member of $\mathfrak{T}$. The same holds for every union of members of $\mathfrak{T}$. Therefore, the set $\mathfrak{T} = \{\emptyset, A, X\}$ is a topology on X. It is called the **topology generated** by A.

In particular, the set $\mathfrak{A} = \{\emptyset, \{1\}, \mathbf{2}\}$ is a topology on $\mathbf{2}$. The topological space $(\mathbf{2}, \mathfrak{A})$ is called the **Sierpinski-space**.

3. On the continuity of the characteristic function

We recall that a function $f : (X, \mathfrak{O}) \to (X', \mathfrak{O}')$ between two topological spaces is **continuous** if, and only if, $f^{-1}(U') \in \mathfrak{O}$ for all $U' \in \mathfrak{O}'$.

Let $(X, \mathfrak{O})$ be a topological space. Then the characteristic function

$$\chi_A : (X, \mathfrak{O}) \to (\mathbf{2}, \mathfrak{A})$$

of a subset A of X is *continuous* if, and only if, $A \in \mathfrak{O}$.

Proof :

Assume that the characteristic function χ_A is continuous. Then $\chi_A^{-1}(\{1\}) \in \mathfrak{O}$, since $\{1\} \in \mathfrak{A}$. But, $\chi_A^{-1}(\{1\}) = A$. Therefore $A \in \mathfrak{O}$.

Conversely, suppose that $A \in \mathfrak{O}$, that is, $\chi_A^{-1}(\{1\}) \in \mathfrak{O}$. Clearly, $\chi_A^{-1}(\emptyset) = \emptyset \in \mathfrak{O}$ et $\chi_A^{-1}(\mathbf{2}) = X \in \mathfrak{O}$. It follows that $\chi_A^{-1}(V) \in \mathfrak{O}$ for each $V \in \mathfrak{A}$. Thus, the characteristic function χ_A is continuous. □

References

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