# An exercise on the characteristic function and the Sierpinski-space

We set $\mathbf{2} = \{0, 1\}$, and recall that the **characteristic function** of a subset $A$ of a set $X$ is the function denoted by $\chi_A$, from $X$ into $\mathbf{2}$, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \backslash A. \end{cases}$$

**(i)** Verify that the function $\gamma : \mathscr{P}(X) \to \mathbf{2}^X$, from the power set of $X$ into the set of all functions of $X$ into $\mathbf{2}$, given by $\gamma(A) = \chi_A$, is a bijection.

**(ii)** Given a set $X$ and a subset $A$ of $X$, show that the set $\mathfrak{T} = \{\emptyset, A, X\}$ is a topology on $X$. In particular, the set $\mathfrak{A} = \{\emptyset, \{1\}, \mathbf{2}\}$ is a topology on $\mathbf{2}$. The topological space $(\mathbf{2}, \mathfrak{A})$ is called the **Sierpinski-space**.

**(iii)** Let $(X, \mathfrak{O})$ be a topological space. Prove that the characteristic function

$$\chi_A : (X, \mathfrak{O}) \to (\mathbf{2}, \mathfrak{A})$$

of a subset $A$ of $X$ is *continuous* if, and only if, $A \in \mathfrak{O}$.